

CATCHMENT SCALE COMPARISON ON LUMPED REPRESENTATION FOR A DISTRIBUTED SEDIMENT RUNOFF MODEL

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ABSTRACT

This study analyses catchment scale effect to a lumped sediment runoff model performance derived from lumping a physically-based distributed sediment runoff model. The proposed lumped model takes into account the different sources of erosion and deposition processes on hillslopes. The eroded soils provided to river channel flow with sediment transport mechanism. To investigate catchment scale dependency of the new lumped sediment runoff model, sediment runoff simulations are conducted using different sizes of the catchment area and spatially averaged hourly rainfall. Four different sizes of catchment area are used, representing low-, medium-, and high- resolution. Then, the lumped model is installed for each catchment size scenario. The proposed method is examined by comparing water and sediment discharges simulated by the lumped and the original distributed sediment runoff models in the Lesti River basin, East Java, Indonesia.

Keywords: Lumping, catchment scale, sediment runoff model, Lesti River basin.

1. INTRODUCTION

The Lesti River catchment (381.2 km²), a tributary catchment in the Upper Brantas River basin is selected as study area. The Brantas River, 320 km length with 39 tributaries and a catchment area about 11,800 km², is the second largest river located in East Java, has an average annual rainfall 2000 mm (80% fall in rainy season). At the confluence point of the Lesti River and the Brantas main reach, the Sengguruh dam was constructed in 1988 for water resources and power generation, which has the effect of trapping sediment into the Sutami Dam (formerly Karangates) from the Brantas River and the Lesti River. The Sengguruh Dam has a catchment area of 1,659 km². Its original gross storage in 1989 was 22.4 million m³ and reduced to 5.5 million m³ in 1993. Unexpectedly, most of the gross storage has been already filled with the large amount of sedimentation from the Lesti River (Takara et al., 1996). The Lesti River transports sediment derived from the lahar (volcanic ashes and sands) of Mountain Semeru.

In order to effectively protect and manage water resources there is need to develop the science and to assemble the necessary information on which to base decision making. Herein, estimations of the changes in total runoff and sediment yield as well as an

understanding of those processes mechanism with time and space in the catchment scale are quite important for solution of a number of problems. Design and operational of dams and reservoirs, design of soil conservation, land-use planning, water quality and aquatic habitat management are some of the examples.

Mathematical models, including physically-based, can help to make quantified evaluation, prediction, or to understand the important processes and interactions in sediment runoff phenomenon. Physically-based distributed sediment runoff model and it lumping for Lesti River catchment were newly developed by authors (Apip et al., 2008 and Takara et al., 2001). The main propose to lump a distributed model is to produce a new lumped sediment runoff model version as interest in sediment runoff modelling extends to large catchments scale and to reduce computational time. Governing lumped model parameters derived physically by keeping the physical meanings of an original distributed model, which are obtained from integration of distributed equation according to lumping distributed approach and catchment characteristic from grid-cell based scale to the catchment scale, then new type of lumped model version is run without any additional calibration.

Earlier research results have shown that the rainfall runoff simulations from a well-validated hydrological model scale dependent to catchment area and the complex patterns. The lumped scale for rainfall runoff model is about 200 km², however the lumped scale of sediment runoff model is not clear. This paper illustrates the relation between catchment area, water discharge (total runoff), and sediment concentration by a combination of simulated of those values by an original distributed sediment runoff model and it lumped at different spatial scales inside study area.

2. DISTRIBUTED SEDIMENT RUNOFF MODEL

The physically-based distributed sediment runoff model has been developed to determine the runoff hydrograph, sedimentgraph, and total sediment runoff generated from any temporally-spatially varied rainfall event and continuous rainfall data input. The modeling approach is deterministic, physically-based, empirical, spatially distributed and dynamical in time. Dynamic spatial of water movements, erosion patterns and sediment rates can be predicted at any location inside the catchment as well. The concept of physically-based distributed sediment runoff modeling is shown in Figure 1. A sediment transport algorithm is newly added to the rainfall runoff model. Sediment runoff simulation can be divided in two parallel phases: runoff generation and soil detachment.

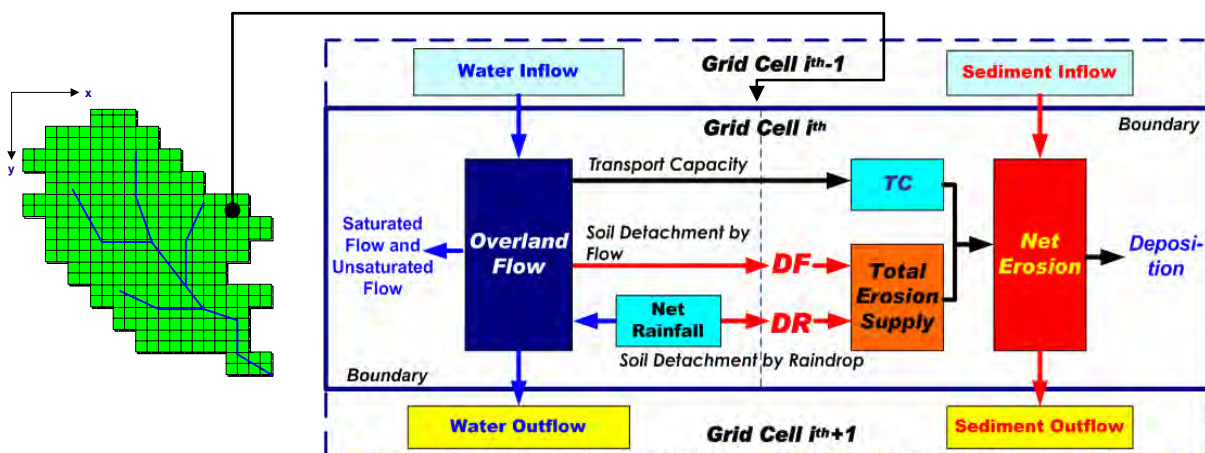


Figure 1 Schematic diagram of the physically-based distributed sediment runoff model within grid-cell scale.

For a given rainfall event, once the rainfall is directly added to subsurface or surface flow according to the water depth on the rainfall dropping grid-cell. The model does not consider the initial rainfall losses due to vertical water flow directly, such as infiltration effects. Rainfall runoff model effectively simulates lagged subsurface flow with calibrated hydraulic conductivities and soil layers depth.

The hydrological model considers three principal water flux pathways within a catchment: subsurface flow through unsaturated flow (capillary pore), subsurface flow through saturated flow (non-capillary pore), and surface overland flow. Using a stage-discharge relationship, after the water depth is greater than the surface soil layer, the net rainfall will accumulate as surface water and begin to flow as overland flow. Subsurface and surface flows in both land surface and river channel networks are computed as kinematic wave. The eroded sediment is transported by overland flow to river channels.

The sediment transport algorithm includes multiple sources of sediment transport, which are soil detachment by raindrop (*DR*) and hydraulic detachment or deposition driven by overland flow (*DF*). Soil detachment for interrill and rill implicitly are simulated respectively, rain splash and flow detachment. The erosion or deposition rates are calculated as a function of the hydraulic properties of the flow, the physical properties of the soil and the surface characteristic. The detachment of soil particles by raindrop impact is function of the energy imparted to the soil surface by the individual drops. The basic assumption of this model is that the sediment is transported and yielded when overland flow occurs. The transport capacity of the overland flow also needs to be specified, in which suspended sediment flow is calculated using the transport capacity approach, as it acts as an upper limit to the potential contribution of each grid-cell to sediment concentrations in saturated areas.

The simulation area is divided into an orthogonal matrix of square cells (250mx250m), assumed to represent homogenous conditions according to the digital elevation model (DEM). This allows the use of DEM to derive flow direction map to define the interaction between the objects which simulate sediment runoff at each grid-cell. Runoff generation, soil erosion or deposition are computed for each grid-cell and are routed between grid-cells using the kinematic wave model following water flow direction, which defines the routine order for the water flow and sediment transport propagation. The model uses the one-dimensional kinematic wave equation for both subsurface and surface flow.

3. LUMPING OF DISTRIBUTED SEDIMENT RUNOFF MODEL

Lumped Sediment Runoff Model Based on Traditional Method

A simple model of catchment response by separating hillslope process and river channel process (Sivapalan et al., 2002) is adopted and extended to incorporate sediment transport processes. This study uses the same principle to explore how sediment yield is related to hydrological response, erosion source, transport mechanism and depositional processes.

On a rainfall event basis, the sediment runoff processes are assumed only affected by surface runoff without consider the effect of sediment load from subsurface layer. According with storage-type concept, the model consists of three water stores, it called rainfall runoff model, and two sediment stores, it called sediment runoff model (Figure 2).

When the water depth larger than the maximum subsurface flow depth of Tank 1, surface runoff occurs and is added to Tank 2, outflow discharge (Q_w) from Tank 3 as a function of water storage amount (S_w) from each Tanks. After overland flow occurs, sediment transport mechanism on hillslope is computed (Tank 2). The sediment storage (S_s) in (Tank 2) is supplied by the balance between hillslope soil erosion rate, redeposition rate,

and sediment discharge released to river channel. Herein, soil erosion by effective rainfall (DR), soil erosion or redeposition by overland flow (DF) are calculated.

Similarly, at a given time t , the river channel store in Tank 3 is supplied with sediment material from the hillslope plus the river bed erosion, and only suspended bed material load is considered and soil detachment due to rain drop energy is neglected. Some of total sediment load, as amount of wash load plus suspended material load, in river channel store is redeposited back into river bed, while another fraction is transported to catchment outlet. Similarly with the hillslope process, the mass of sediment stored in the river channel is determined by the balance between hydraulic erosion rate, redeposition rate, and the release of sediment discharge to the catchment outlet. The rate of erosion or redeposition is depending on the transport capacity of flow and current sediment concentration carried by flow.

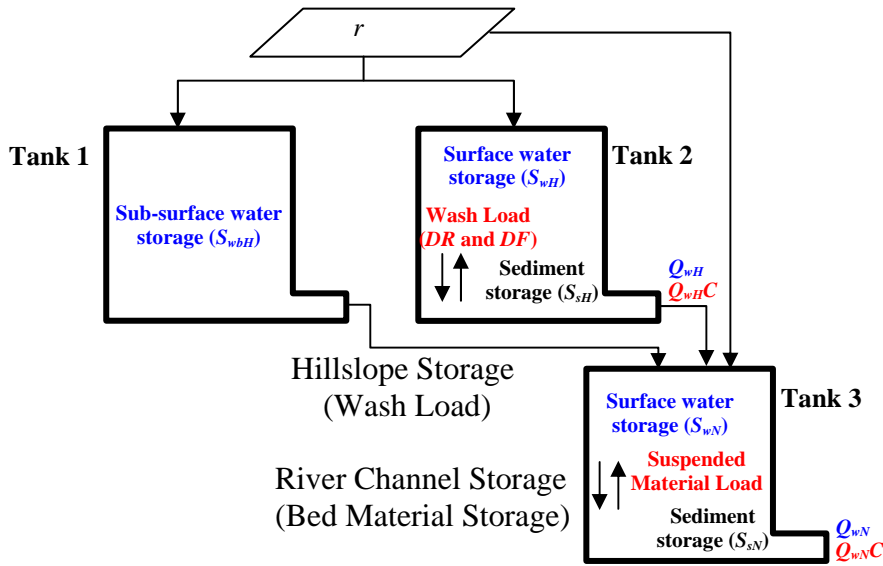


Fig. 2 Schematic diagram of the lumped sediment runoff model at catchment scale

The continuity equation of runoff and sediment models is represented as follows:
Hillslope process:

$$\frac{dS_{wH}}{dt} = r_H A_H - Q_H \quad (1)$$

$$\begin{aligned} \frac{dS_{sH}}{dt} &= DR + DF_H - Q_H C_H \quad (2) \\ &= k 56.48 r_H e^{-b h_{s-avr}} + (\alpha (S_{sH}^{\max} - S_{sH}) h_{s-avr} - Q_H C_H 3600) / A_H \end{aligned}$$

River channel process:

$$\frac{dS_{wN}}{dt} = r_N A_N + Q_H - Q_N \quad (3)$$

$$\begin{aligned} \frac{dS_{sN}}{dt} &= Y_H + DF_N - Q_N C_N \quad (4) \\ &= Q_H C_H + (\alpha (S_{sN}^{\max} - S_{sN}) h_{s-avr} - Q_N C_N 3600) / A_N \end{aligned}$$

where k is the soil detachability (kg/J); KE is the total kinetic energy of the net rainfall (J/m^2); and b is an exponent to be tuned; h_{s-avr} is the average of overland flow water depth at the

hillslope/river channel, Y_H is the hillslope sediment yield; α is the erosion/ deposition efficiency factor; r is the effective rainfall intensity; S_s^{max} is the maximum storage amount of sediment concentration; A is the total area; and the subscript of H and N show the hillslope and river channel section, respectively.

If the effective rainfall intensity is known, Eqs. (1,2,3,4) cannot be solved directly to obtain the outflow hydrograph and sedimentgraph from hillslopes or river channels, because other variables are unknown. A relationship is needed to relate Q , S_w , S_{ws} , h_{s-avr} , and S_s^{max} which are estimated from lumping distributed approach.

Lumping Distributed Rainfall Runoff Model

As first stage of lumped model development, a method to lump a distributed sediment runoff model for one layer, in case the sub-surface layer was assumed reached to saturated condition, was derived (Apip et al., 2007) as an extension from the lumping method proposed by Ichikawa et al., (2000). Herein, lumped rainfall runoff model derived from lumping distributed rainfall runoff model under steady-state condition is expressed by a non-linear reservoir, the storage is non-linearly related to outlet water discharge by storage constants K and p as follows:

$$S_w = K Q^p \quad (5)$$

by substituting Eq. 5 into Eq. 1 or Eq. 3 becomes:

$$\frac{dS_{wN}}{dt} = r_N A_N + Q_H - \left(\frac{S_w}{K}\right)^{1/p} \quad (6)$$

K is the model parameter having a physical meaning, can be interpreted as the time of concentration for a kinematic wave to travel across the system.

The value of K is derived from the lumping distributed rainfall runoff approach (Apip et al., 2007). K is influenced by spatially distributed of slope length (L), slope gradient (i), roughness coefficient (n), upper contributing area (U), and total area (A). It proves that K can be derived from the integration of distributed equation. In new lumped model, K is dimensional parameter ($m^{6/5} s^{3/5}$) is defined as:

$$K = \sum_{j=1}^N \frac{w}{(A)^p} \frac{k_j}{p+1} \left(\left(L_j + \frac{U_j}{w} \right)^{p+1} - \left(\frac{U_j}{w} \right)^{p+1} \right) \quad (7)$$

in which $p = \frac{1}{m}$, $k = \left(\frac{1}{\alpha} \right)^{\frac{1}{m}}$, and $\alpha = \frac{\sqrt{i}}{n} = \frac{\sqrt{\sin \theta}}{n}$

where i is the slope gradient (m/m), n is the roughness coefficient, m is the exponent constant, which can be shown to be 5/3 from manning's equation, and j is the number of grid-cell. Discharge per unit width (q), flow velocity (v), and water storage (s_w) for each grid-cell defined as:

$$q_i = \bar{r} \left(\frac{U_i}{w} + x_i \right), \quad h_i = \left(\frac{\bar{r} \left(\frac{U_i}{w} + x_i \right)}{\alpha_i} \right)^{\frac{1}{m}}, \quad v_i = \left(\frac{Q}{A} \left(L_i + \frac{U_i}{w} \right) \right)^{1-p} k_i^{-1}$$

$$s_{wi} = w \left(\frac{\bar{r}}{\alpha_i} \right)^{\frac{1}{m}} \frac{1}{\frac{1}{m} + 1} \left(\left(L_i + \frac{U_i}{w} \right)^{\frac{1}{m} + 1} - \left(\frac{U_i}{w} \right)^{\frac{1}{m} + 1} \right)$$

Total water storage at the hillslope area or river channel area are expressed as:

$$S_w = \sum_{i=1}^N S_{wi}$$

Lumping Distributed Sediment Runoff Model

The sediment runoff processes in this study are affected by dynamic spatial distribution of overland flow. The relationship between detachment and redeposition represented by Eq. 2 and Eq. 4 depends on the balance between S_s and S_s^{max} , the depth of overland flow or total runoff as well as. Those variables are produced from lumping distributed approach as:

The Maximum Sediment Storage (S_s^{max})

Many, mostly empirical, equations have been developed to predict sediment transport capacity of flow (TC) as function flow characteristics, slope, and material characteristics. These equations often use a threshold value of stream power, shear stress, or discharge. In this study, the transportation capacity is calculated based on the Unit Stream Power (USP) theory that can be applied for sediment transport in open channels and surface land erosion (Yang, 1973). The USP theory stems from a general concept in physics that the rate of energy dissipation used in transporting sediment materials should be related to the rate of material being transported. Sediment concentration in the water flow must be directly related to USP. The USP theory contributing to TC is defined as a product of the overland flow velocity, v , and slope, i , in the i^{th} grid-cell. Small particles such as clay and silt move mostly in suspension and easily carried by the flow while the sand fraction moves as bed-material and more difficult to move by flow. This is accomplished that TC depends on the particle settling velocity, shear velocity, grain size, kinematic viscosity of the water, and water density. A relationship between USP and the upper limit to the sediment concentration in the overland flow, C_t (ppm), can be derived (Yang, 1973 and Yang, 1979). Hence TC is the product of C_t as:

a. Surface Land and Erosion:

$$TC = \log C_t = I + J \log((vi - v_{critical}i) / \omega) \quad (8)$$

in which:

$$I = 5.435 - 0.386 \log(\omega D_{50} / NU) - 0.457 \log(U^* / \omega)$$

$$J = 1.799 - 0.409 \log(\omega D_{50} / NU) - 0.314 \log(U^* / \omega)$$

$$\omega = \sqrt{\frac{2}{3} + \frac{36}{\left(\frac{\rho_s}{\rho_w} - 1\right) g \left(\frac{D_{50}}{1000}\right)^{2/NU}}} - \sqrt{\frac{36}{\left(\frac{\rho_s}{\rho_w} - 1\right) g \left(\frac{D_{50}}{1000}\right)^{2/NU}}}$$

where vi is the unit stream power, m/s (v is flow velocity in m/s and i is the slope gradient m/m); $v_{critical}i$ is the critical unit stream power ($v_{critical}$ is the critical flow velocity); ω is the sediment fall velocity (m/s) calculated by Rubey's equation; ρ_s is the sediment particle density (kg/m^3); ρ_w is the water density (kg/m^3); g is the specific gravity (m/s^2); D_{50} is the median of grain size (mm); and NU is the kinematic viscosity of the water (m^2/s). $U^*(=\sqrt{g i h_s})$ is the average shear velocity (m/s).

b. River Channel Erosion

The sediment transport function within river channel has been intended for the estimation of sediment transport rate or concentration at a nonequilibrium condition with deposition process. When the wash load and concentration of fine material is high, nonequilibrium bed-material sediment transport may occur, and its amount is a function of wash load. Wash load which depends on the supply from hillslopes has been assumed is high enough to significantly affect the fall velocity of sediment particles, flow viscosity, relative density of sediment and water.

For flow in river channel and at a nonequilibrium, transport capacity concentration of flow is modeled as a function of modified Yang's unit stream power, which is an expression for the total load with high concentration of fine sediment particle. Regarding Eq. 8 when sediment concentration is not too low, the incipient motion criteria, called critical stream power, can be neglected. To apply Eq. 8 to a river with a high concentration of fine materials and wash load, the values of viscosity, fall velocity, and relative density have to be modified to consider the influence of high concentration of fine material on those values. Herein, the modified unit stream power formula proposed by Yang et al. (1979) is expressed:

$$TC = LogC_i = 5.165 - 0.153 \frac{\omega_m D_{50}}{NU_m} - 0.297 \log \frac{U_*}{\omega_m} + (1.780 - 0.360 \log \frac{\omega_m D_{50}}{NU_m} - 0.480 \log \frac{U_*}{\omega_m}) \log \left(\frac{\gamma_m}{\gamma_s - \gamma_m} \frac{v_i}{\omega_m} \right) \quad (9)$$

the coefficients in Eq. 9 are identical to those in Eq. 8. However, the values of ω , γ_s , and NU are modified for sediment transport in sediment-laden

The maximum sediment storage on hillslopes or river channels scale is defined as the total sediment transport capacity of overland flow in a whole of the hillslopes for each time step calculation. Therefore, we expressed the maximum sediment storage as the function of TC from the i^{th} grid-cell, surface water storage amount in the i^{th} grid-cell (s_{wsi}), and S_{ws} . The maximum sediment storage at hillslope area is calculated by adding up TC_i multiplied to s_{wsi} for all grid-cells as:

$$S_s^{\max} = \sum TC_i s_{w_i} / (S_w 1000) \quad (10)$$

TC (ppm) for the i^{th} grid-cell has been estimated by:

Sediment Concentration (C)

Based on the relationship between the current sediment storage (S_s) ($\text{kg}/\text{m}^3/\text{hr}$) and S_w for each time step calculation, the value of C from hillslope/river channel area can be solved:

$$C = \frac{S_s}{S_{ws}} \quad (11)$$

for each time-step calculation C is assumed to be uniform over the hillslope/river channel area and this is the variable of sediment continuity (see Eqs. 2 & 4).

4. COMPARISON OF LUMPED AND DISTRIBUTED MODEL UNDER DISCRETIZATION OF CATCHMENT AREA

The numerical experiments were run in two cases of rainfall event mode with hourly rainfall data input (Case 1: short duration, and Case 2: long duration) at four cases catchment

size. A digital topographic model for the Lesti River catchment was first developed and then four scenarios of the catchment size (Case A, Case B, Case C, and Case D), the high-, medium-, and low resolution of catchment size in term of lumped mechanism were delineated (see Figure 3) and its characteristics are given in Table 1. For each catchment size, average rainfall over whole area was used.

The original distributed sediment runoff model parameters were calibrated and validated using historical data for the 351.3 km² of the Lesti River catchment. These calibration and validation runs suggest that observed data of hydrological condition and sediment transport mechanism is generally amenable to the hydrological and sediment transport mechanism based on the original distributed sediment runoff model. The lumped model algorithm was applied to each synthetic of catchment area and compared with simulation results using distributed model.

Table 1 The characteristic of four cases catchment size in the Lesti River catchment

Catchment	Total grid-cell	Area (km ²)
Case A	404	25.3
Case B	1743	108.9
Case C	2114	132.1
Case D	6101	351.3

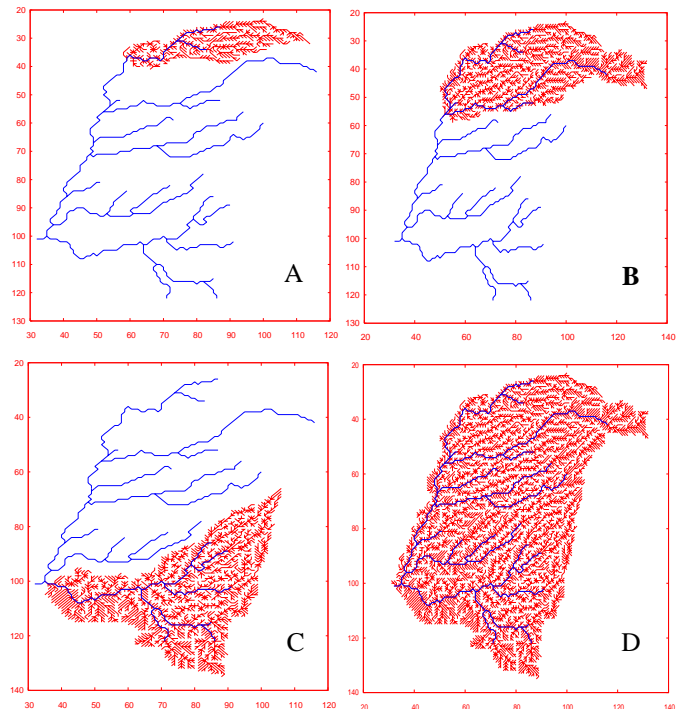


Figure 3 Lesti River catchment was partitioned into four synthetic catchment area, in which red lines are the catchment area and blue lines are river channels.

Figure 4 is plots of the water discharge and sediment concentration by the physically-based distributed sediment runoff model and the lumped model for the entire period of rainfall scenario (Case 1 and Case 2). Figure 4 shows the discrepancy of simulation results between the original distributed model and its lumped model version for both water discharge and sediment concentration. The lumped model is in an acceptable way to reproduce the discharge and sediment concentration simulated by distributed model. All the values of correlation coefficient more than 0.90, its mean that the lumped model is in an acceptable way to reproducing the water discharge and sediment concentration calculated by distributed model. Just as with the case of sediment concentrations the correlation coefficients lower than outflow discharges.

However, in the case water discharge when the catchment size is increase, discrepancy between simulated values by the lumped and distributed models is generally higher. In the case of simulated sediment concentrations, the differences are higher than simulated water discharges, variation depending on the catchment size and characteristics as well as.

The discrepancy between lumped and distributed models for water discharges simulated in Case D under the rainfall condition Case 1 is higher than Case 2, in which the lumped model tends to underestimate, is due to the assumed steady state condition in deriving

lumped model, total rainfall in Case 1 is less than total rainfall of Case 2.

Conventionally sediment yield is held to decrease as basin area increases. For drainage areas in the range between small catchments and large basin ($>10\text{km}^2$), the effect of sediment sinks often becomes dominant over sediment sources, resulting in a gradual decline in sediment yield. The explanations for this is that large catchments often have more extensive floodplain development as well as footslopes where sediment are stored, the travel distance for sediment through catchment is longer and small catchments are more likely to respond directly to event driven flood than large catchments. The characteristics of sediment yield simulated by lumped model are consistent with these studies (see Case A).

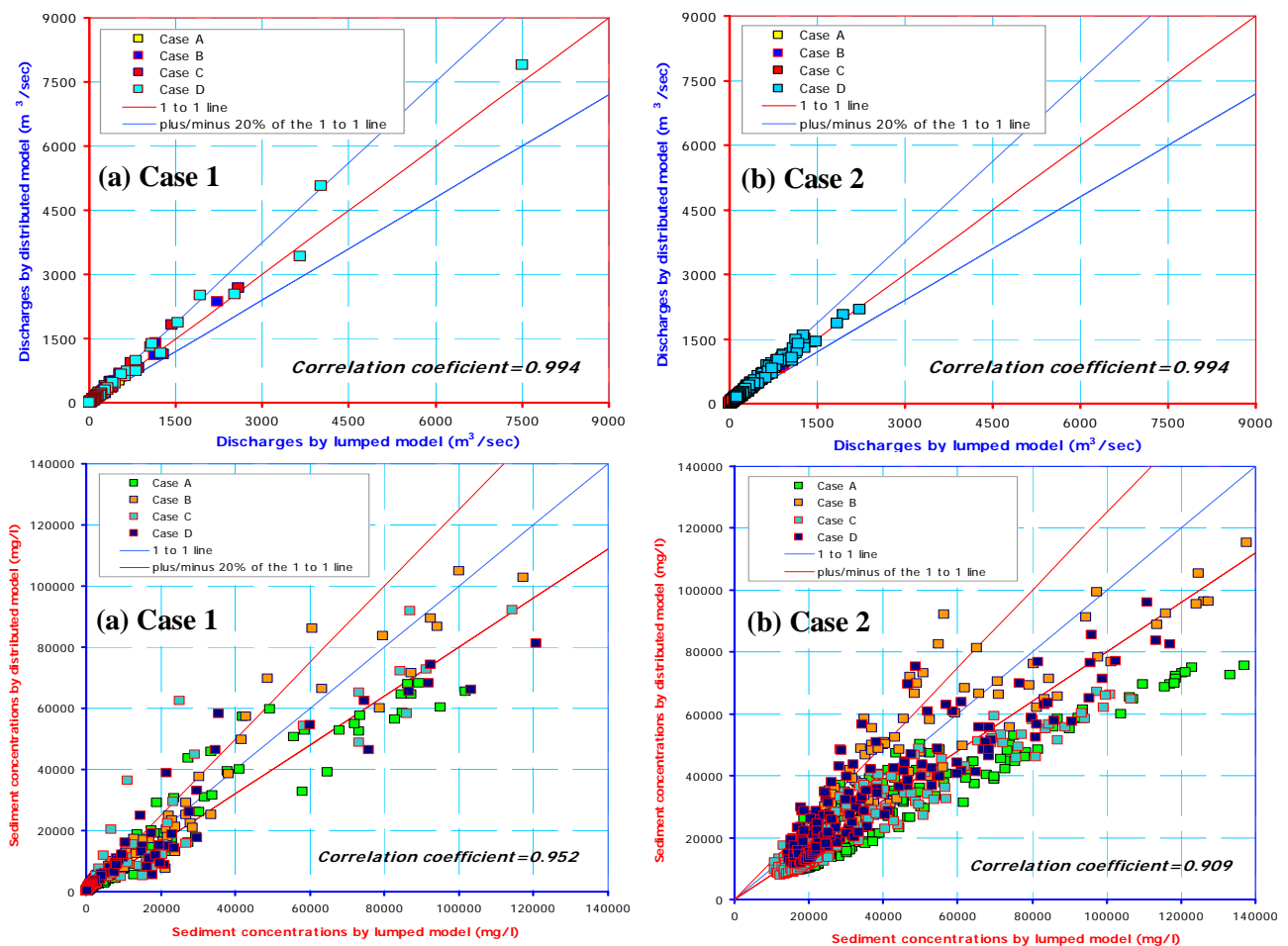


Figure 4 Comparison of simulated outflow discharges and sediment discharges by lumped and distributed models for two scenario rainfall data input: (a) Case 1; and (b) Case 2 and combined with four scenarios of synthetic catchment area.

CONCLUSIONS

Within the range of possible catchment area and soil depth, simulation results computed by the lumped model for hillslope area and river channel agree well with the simulation results computed by the original distributed model. For lumped rainfall runoff model, the discrepancy of the lumped model and distributed model increases when the catchment area and soil thicknesses increase, low accumulative rainfall amount, and/or spatial temporal variation of rainfall are large. In the case of simulated sediment concentrations, the

differences are higher than simulated water discharges, variation depending on the catchment size and characteristics as well as.

The analyses spatial scale dependency of a lumped sediment runoff model derived from a physically-based distributed sediment runoff model under land use and rainfall scenario and its application for large catchments are important areas of further research.

ACKNOWLEDGMENTS

This study was supported by the MEXT Coordination Fund for Promotion of Science and Technology, Japan Science and Technology Agency (PI: Prof. Kaoru Takara, DPRI, Kyoto University).

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